

Notes on wormhole existence in scalar-tensor and $F(R)$ gravity

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Some recent papers have claimed the existence of static, spherically symmetric wormhole solutions to gravitational field equations in the absence of ghost (or phantom) degrees of freedom. We show that in some such cases the solutions in question are actually not of wormhole nature while in cases where a wormhole is obtained, the effective gravitational constant G_{eff} is negative in some region of space, i.e., the graviton becomes a ghost. In particular, it is confirmed that there are no vacuum wormhole solutions of the Brans-Dicke theory with zero potential and the coupling constant $\omega > -3/2$, except for the case $\omega = 0$; in the latter case, $G_{\text{eff}} < 0$ in the region beyond the throat. The same is true for wormhole solutions of $F(R)$ gravity: special wormhole solutions are only possible if $F(R)$ contains an extremum at which G_{eff} changes its sign.

1 Introduction

It is well known that to build a static traversable wormhole in general relativity it is necessary to invoke a matter source of gravity that violates the Null Energy Condition (NEC), at least in the neighborhood of the wormhole throat [1]. With dynamic wormholes the situation is more complex: first, the notion of a wormhole throat is then less evident and even admits different definitions [2, 3]; second, a dynamic wormhole may exist not eternally but only in a certain time interval, and in this case the requirements to its matter source may be weakened [4]. In what follows, we will restrict ourselves to static wormhole space-times.

The nonexistence theorem for static wormholes in the presence of any matter respecting the NEC was recently generalized [5] to the class of theories of gravity whose space-times are related to that of general relativity by a conformal mapping. This class includes theories without ghost fields even though many of them admit NEC violation. The generalization occurs under certain conditions. Thus, for a scalar-tensor theory (STT) of gravity, formulated in a space-time \mathbb{M}_J (the Jordan frame) with the metric $g_{\mu\nu}$ using the Lagrangian

$$L_{\text{STT}} = \frac{1}{2} \left[f(\Phi)R + h(\Phi)g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} - 2U(\Phi) \right] + L_m \quad (1)$$

(R is the Ricci scalar, L_m is the matter Lagrangian, f , h and U are arbitrary functions), the above

theorem holds if the non-minimal coupling function $f(\Phi)$ is everywhere positive (in other words, the graviton is not a ghost) and also

$$l(\Phi) := fh + \frac{3}{2} \left(\frac{df}{d\Phi} \right)^2 > 0 \quad (2)$$

(the Φ field itself is not a ghost).² The latter condition becomes evident if one performs the standard conformal mapping to the Einstein frame (the space-time \mathbb{M}_E with the metric $\bar{g}_{\mu\nu}$) such that

$$\bar{g}_{\mu\nu} = |f(\Phi)|g_{\mu\nu}, \quad (3)$$

after which the Lagrangian (1) transforms to

$$2L_E = (\text{sign } f)[\bar{R} + \varepsilon\bar{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu}] - 2|f|^{-2}U + 2|f|^{-2}L_m, \quad (4)$$

where $\varepsilon = \text{sign } l(\Phi)$, bars mark quantities obtained from or with $\bar{g}_{\mu\nu}$, indices are raised and lowered with $\bar{g}^{\mu\nu}$ and $\bar{g}_{\mu\nu}$, and the Φ and ϕ fields are related by

$$\frac{d\phi}{d\Phi} = \frac{\sqrt{|l(\Phi)|}}{f(\Phi)}. \quad (5)$$

The Brans-Dicke (BD) STT is the special case of (1) corresponding to

$$\begin{aligned} f(\Phi) &= \Phi, & h(\Phi) &= \omega/\Phi, \\ l(\Phi) &= \omega + 3/2, & \omega &= \text{const.} \end{aligned} \quad (6)$$

²Our conventions are: the metric signature $(+ - - -)$; the curvature tensor $R^\sigma{}_{\mu\rho\nu} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \dots$, $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$, so that the Ricci scalar $R > 0$ for de Sitter space-time and the matter-dominated cosmological epoch; the system of units $8\pi G = c = 1$.

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Let us recall that the NEC reads $T_{\mu\nu}k^\mu k^\nu \geq 0$, where $T_{\mu\nu}$ is the stress-energy tensor (SET) of matter and k^μ is an arbitrary null vector with respect to the metric $g_{\mu\nu}$. The mapping (3) transforms the SET according to $\bar{T}_{\mu\nu} = |f|^{-1}T_{\mu\nu}$ while k^μ remains a null vector in the metric $\bar{g}_{\mu\nu}$. Thus if the NEC holds in \mathbb{M}_J , it also holds in \mathbb{M}_E , and vice versa, and it means, in particular, that wormholes are absent in \mathbb{M}_E .

Assuming that matter in \mathbb{M}_E does respect the NEC, the above theorem is proved in [5] (following the idea expressed earlier in [6]) using the fact that if both $f(\Phi)$ and $l(\Phi)$ are smooth and positive everywhere, including limiting points, the mapping (3) transfers a flat spatial infinity in one frame to a flat spatial infinity in the other. Therefore, if we suppose that there is an asymptotically flat wormhole in \mathbb{M}_J , its each flat infinity has a counterpart in \mathbb{M}_E , the whole manifold \mathbb{M}_E is smooth, and we obtain a wormhole there, contrary to what was assumed. We conclude that static and asymptotically flat wormholes are absent in the Jordan frame as well.

A special case of this situation is connected with matter concentrated on a thin shell. Accordingly, as was explicitly shown in [7], in any STT with a non-ghost scalar field, in *any* thin-shell wormholes built from two identical regions of static, spherically symmetric space-times, the shell has negative surface energy density, thus violating both null and weak energy conditions.

In [5], a possible wormhole behavior of static, spherically symmetric configurations was also considered in STT under weakened requirements, allowing $f(\Phi)$ to reach zero or even become negative. It has turned out that if f only reaches zero, twice asymptotically flat wormhole solutions in the Jordan frame can exist but only in exceptional cases: (i) the corresponding Einstein-frame solution must comprise an extreme black hole, whose double horizon is then mapped to the second spatial infinity in \mathbb{M}_J ; it is not possible with vacuum solutions but can happen with nonzero electric or magnetic fields;³ (ii) additional fine tuning is necessary to avoid a solid angle deficit or excess at this second infinity, and (iii) the theory itself should be very

special.

Rather a wide (although still special) class of wormhole solutions exists in theories where a transition to $f < 0$ is allowed, so that the manifold \mathbb{M}_E is mapped according to (3) to only a part of the manifold \mathbb{M}_J (the conformal continuation phenomenon [10, 11]). However, previous studies have shown that such solutions are generically unstable under spherically symmetric perturbations [12], the instability appearing due to a negative pole of the effective potential at the transition surface to $f < 0$. The existence of this pole still does not guarantee an instability, and a further study of non-linear dynamical evolution is necessary; but even if such wormholes might exist, their “remote mouths” would be located in anti-gravitational regions with $f < 0$. So, they cannot connect different parts of our Universe but could only be bridges to other universes (if any) with very unusual physics. In addition, it is known that a generic space-like anisotropic curvature singularity arises dynamically if $f \rightarrow 0$ [13], and it is unclear how to avoid it.

Some recent publications contain results on static, spherically symmetric wormhole existence which seem to contradict these conclusions. In particular:

1. It is claimed that there are vacuum wormhole solutions in the BD theory corresponding to the coupling constant ω in the non-ghost interval $(-3/2, -4/3)$ [14].
2. A similar claim is made for the BD theory with ω in a larger non-ghost interval including the value $\omega = 0$ [15].
3. It is asserted that vacuum BD wormholes exist for $\omega < -2$ but nothing is said about the range $-2 \leq \omega < -3/2$ [14].
4. Wormhole solutions are found in some $F(R)$ theories of gravity [16] which are equivalent to the BD theory with $\omega = 0$ and a nonzero potential.

The purpose of this paper is to clarify the situation in all these cases. To this end, we begin the next section with designating the conditions under which a static, spherically symmetric metric is said to describe a wormhole. Then we explicitly write the vacuum solution of the general massless STT and specify its properties for the BD theory, thus

³Such an example has been found in [8]. In a more general context, inclusion of an electric field as a source of gravity in STT enlarges the number of classes of solutions but, just as is the case with vacuum solutions, wormholes can exist either with $\varepsilon = -1$ or in special cases with G_{eff} somewhere becoming infinite or negative [9].

covering items 1–3 above. Section 3 is devoted to the properties of wormhole solutions in $F(R)$ theories. Our previous conclusions [5] are confirmed in all these cases. Section 4 contains some remarks of methodological nature.

2 Wormholes in scalar-tensor gravity

2.1 The wormhole notion

The general static, spherically symmetric metric can be written as

$$ds^2 = e^{2\gamma(u)} dt^2 - e^{2\alpha(u)} du^2 - r^2(u) d\Omega^2. \quad (7)$$

where u is an arbitrary radial coordinate and $d\Omega^2 = (d\theta^2 + \sin^2\theta d\varphi^2)$ is the linear element on a unit sphere. The metric (7) is asymptotically flat as u tends to some value u_0 (finite or infinite) if

$$\begin{aligned} r \rightarrow \infty, \quad \gamma &\approx \gamma_0 + \gamma_1/r + o(1/r), \\ e^{-2\alpha} r'^2 &\rightarrow 1, \end{aligned} \quad (8)$$

where the prime denotes d/du , $\gamma_0, \gamma_1 = \text{const}$, and the last condition in (8) is the requirement of a correct circumference to radius ratio for large circles.

We will say that the metric (7) describes a wormhole if it regular in some range (u_1, u_2) of the radial coordinate, does not contain horizons (that is, $\gamma(u)$ and $\alpha(u)$ are finite) in this range, and is asymptotically flat both as $u \rightarrow u_1$ and as $u \rightarrow u_2$. The existence of two large r asymptotic regions inevitably means that there is at least one regular minimum of the function $r(u)$, called a throat.

We thus consider only globally regular configurations. One could admit more general regular asymptotic behaviors, e.g., anti-de Sitter, but for our present discussion it is sufficient to be restricted to asymptotic flatness. We thus also discard possible wormholes in which $r(u)$ reaches large (as compared with the throat radius) but finite values of r at one or both sides, as happens, e.g., when a wormhole connects two closed worlds. Our arguments could be easily modified to include such cases. Another case of interest which is not covered by our definition and deserves a separate study is that of a cosmological horizon located far enough from the throat, as is the case in some known models [17, 18].

2.2 Vacuum solutions of a general STT

Let us now consider the theory (1) assuming $f \geq 0$, $U \equiv 0$ and $L_m = 0$. Then in the Einstein frame we have a massless, minimally coupled scalar field as the only source of gravity. The static, spherically symmetric solutions to the field equations are well known in this case: these are the Fisher solution of 1948 [19]) if $\varepsilon = +1$ and the Bergmann–Leipnik solution of 1957 [20]) (sometimes called the “anti-Fisher” solution) in case $\varepsilon = -1$. Both solutions were repeatedly rediscovered afterwards. Let us reproduce them in the simplest joint form, following [10].

Two combinations of the Einstein equations for the metric (7) and $\phi = \phi(u)$ read $R_0^0 = 0$ and $R_0^0 + R_2^2 = 0$. Choosing the harmonic radial coordinate u , such that $\alpha(u) = \gamma(u) + 2 \ln r(u)$, we easily solve these equations. Indeed, the first of them reads simply $\gamma'' = 0$, while the second one is written as $\beta'' + \gamma'' = e^{2(\beta+\gamma)}$. Solving them, we have

$$\begin{aligned} \gamma &= -mu, \\ e^{-\beta-\gamma} &= s(k, u) := \begin{cases} k^{-1} \sinh ku, & k > 0, \\ u, & k = 0, \\ k^{-1} \sin ku, & k < 0. \end{cases} \end{aligned} \quad (9)$$

where k and m are integration constants; two more integration constants have been suppressed by choosing the zero point of u and the scale along the time axis. As a result, the metric has the form [10]

$$ds^2 = e^{-2mu} dt^2 - \frac{e^{2mu}}{s^2(k, u)} \left[\frac{du^2}{s^2(k, u)} + d\Omega^2 \right] \quad (10)$$

(note that flat spatial infinity here corresponds to $u = 0$, and m has the meaning of the Schwarzschild mass). Moreover, the scalar field equation in this gauge reads $\phi'' = 0$, hence

$$\phi = Cu, \quad C = \text{const} \quad (\text{the scalar charge}) \quad (11)$$

without loss of generality. Lastly, due to the $\binom{1}{1}$ component of the Einstein equations (the constraint equation), the integration constants are related by

$$2k^2 \text{sign } k = 2m^2 + \varepsilon C^2. \quad (12)$$

Eqs. (10), (11) describe the Fisher solution in the case $\varepsilon = +1$, hence $k > 0$ due to (12); in the case $\varepsilon = -1$, they give the anti-Fisher solution, in which k can be arbitrary. These solutions are simultaneously the Einstein-frame vacuum solutions

of *any* STT (1) with $U(\Phi) = 0$ and $\text{sign } l(\Phi) = \varepsilon$. Considering (10) as $\bar{g}_{\mu\nu} dx^\mu dx^\nu$ and applying the mapping (3), we easily obtain the solution in the Jordan frame \mathbb{M}_J .

In particular, in the BD theory we have according to (5)

$$\begin{aligned} f(\Phi) &= \Phi = \Phi_0 \exp(\phi/\sqrt{|\omega + 3/2|}), \\ \Phi_0 &= \text{const}, \end{aligned} \quad (13)$$

where $\phi = Cu$ and we can put $\Phi_0 = 1$ without loss of generality.

2.3 The BD theory, $\omega > -3/2$, $\varepsilon = +1$.

For $k > 0$, which is always true if $\varepsilon = +1$ (leaving aside the trivial case $k = 0$ with flat metric), it is helpful to apply the coordinate transformation $e^{-2ku} = 1 - 2k/x$, after which the Jordan-frame solution in the BD theory takes the form

$$\begin{aligned} ds_J^2 &= P^{-\xi} \left[P^a dt^2 - P^{-a} dx^2 - P^{1-a} x^2 d\Omega^2 \right], \\ \Phi &= P^\xi, \quad P(x) \equiv 1 - 2k/x, \end{aligned} \quad (14)$$

where we have redefined the integration constants:

$$\xi = -C/(2k\sqrt{\omega + 3/2}), \quad a = m/k < 1, \quad (15)$$

and the relation (12) passes on to

$$(2\omega + 3)\xi^2 = 1 - a^2. \quad (16)$$

The index J is used to stress that it is the Jordan-frame metric. It is the so-called Brans class I solution [22], written with the explicitly separated conformal factor $1/\Phi$ in the metric; in the square brackets in (14) we have the Fisher metric.⁴

Now, can the solution (14) describe wormholes? To answer this question, we notice that the solution is defined and is regular in the range $2k < x < \infty$ and is asymptotically flat as $x \rightarrow \infty$. It is thus sufficient to check if the other end, $x = 2k$, can be another flat infinity or a regular surface beyond which the solution could be continued.

The quantity $g_{tt} = P^{a-\xi}$ is finite and non-zero at $x = 2k$ (i.e., $P = 0$) only if $a = \xi$, i.e., in the special solution in which $g_{tt} \equiv 1$. On the other hand, $r^2 = x^2 P^{1-a-\xi}$ is infinite at $x = 2k$ if $a +$

$\xi > 1$ and is finite there if $a + \xi = 1$. Thus if $a + \xi > 1$, the surface area of the sphere $x = \text{const}$ has a minimum (i.e., there is a throat) at some intermediate point $x = x_1 > 2k$. (Note that this is a would-be wormhole configuration considered in [14, 15] in the case $a \neq \xi$.) However, in all such cases we have either a naked singularity at $x = 2k$ or a repulsive non-flat asymptotic unattainable for test bodies.

Indeed, one can check that, as $x \rightarrow 2k$ ($P \rightarrow 0$), the Kretschmann invariant $\mathcal{K} = R_{\mu\mu\rho\sigma} R^{\mu\mu\rho\sigma}$ of the metric (14) behaves as $P^{2(a+\xi-2)}$. Hence in case $a + \xi < 2$ we have a naked singularity at $x = 2k$. This happens irrespective of a being equal to ξ or not, even though $g_{tt} = \text{const}$ when $a = \xi$ (in which case $\omega < 0$). Thus all such would-be wormhole configurations have naked singularities.⁵

For $a + \xi \geq 2$ we have a finite (for $a + \xi = 2$) or zero limit of \mathcal{K} at $x = 2k$. The range of ξ required for that is

$$\begin{aligned} \frac{2 - \sqrt{-6\omega - 8}}{2\omega + 4} &\leq \xi \leq \frac{1}{\sqrt{2\omega + 3}}, \\ -3/2 < \omega < -11/8, \\ \frac{2 - \sqrt{-6\omega - 8}}{2\omega + 4} &\leq \xi \leq \frac{2 + \sqrt{-6\omega - 8}}{2\omega + 4}, \\ -11/8 \leq \omega \leq -4/3. \end{aligned} \quad (17)$$

Thus the maximum possible value of ω in this case is $\omega = -4/3$, achieved at $a = 1/2$, $\xi = 3/2$. However, since $a < 1$ and therefore $\xi > a$, we inevitably have $g_{tt} = P^{a-\xi} \rightarrow \infty$, i.e., this asymptotic is repulsive and inaccessible to timelike geodesics. Moreover, the limit $x \rightarrow 2k$ is characterized by an infinite proper distance along the radial direction ($\int P^{-(\xi+a)/2} dx$ diverges). However, the iterated integral of the Riemann tensor components in an orthonormal frame diverges as $x \rightarrow 2k$, just as it happens in the case of usual “strong” singularities, which also indicates the absence of a globally regular behavior there (see also the discussion in [21] in this connection). Summarizing, this configuration cannot be called a wormhole according to our definition. Though, one certainly cannot exclude that its part containing a throat might be used for obtaining a wormhole by, say, a cut-and-paste procedure with some asymptotically flat space-time.

⁴The form (14) of the solution coincides with the one used in [23] if we re-denote $a - \xi = A$, $a + \xi = -B$. The substitution $x = y[1 + k/(2y)]^2$ converts it to the isotropic coordinates employed, e.g., in [14, 22], and the constants C , λ and B used there are related to ours by $C = 2\xi/(a - \xi)$, $\lambda = 1/(a - \xi)$, $B = k/2$.

⁵A similar situation occurs in Einstein gravity when one attempts to construct a would-be wormhole solution supported by the phantom Chaplygin gas [24] or the phantom generalized Chaplygin gas [25]: a curvature singularity arises at a finite value of the spherical radius r .

Let us note that throats appear at all values of $\omega > -3/2$, even very large ones. Indeed, suppose, e.g., $\xi = 2(1-a)$, so that $a + \xi = 2 - a > 1$. Then by (16) we have

$$2\omega + 3 = \frac{1+a}{4(1-a)},$$

which can be made arbitrarily large by taking a close enough to unity. More generically, as follows from Eq. (16), $a + \xi > 1$ if $0 < \xi < (\omega + 2)^{-1}$. But, as we have seen, in all such cases wormholes are not obtained.

Of interest is the special case $a = \xi = 1/2 \Rightarrow \omega = 0$, the only case where the sphere $x = 2k$ is regular. The metric (14) then acquires the so-called spatial Schwarzschild form

$$\begin{aligned} ds_J^2 &= dt^2 - dx^2/(1 - 2k/x) - x^2 d\Omega^2 \\ &= dt^2 - 4(2k + y^2)dy^2 - (2k + y^2)^2 d\Omega^2, \end{aligned} \quad (18)$$

where $y^2 = x - 2k$. It is a wormhole metric, defined in the range $y \in \mathbb{R}$. The initial range $x > 2k$ (in which the manifold \mathbb{M}_E is defined) corresponds to either $y > 0$ or $y < 0$. The Jordan-frame manifold \mathbb{M}_J consists of two regions $y > 0$ and $y < 0$, each of them mapped into \mathbb{M}_E according to (3), plus the regular sphere $y = 0$, the throat.

Thus the only wormhole solution in this family exists for $\omega = 0$ due to the conformal continuation phenomenon [10, 11]. The transition sphere corresponds to $\Phi = 0$; in the whole range of y we have $\Phi = y/\sqrt{y^2 + 2k}$, so that at $y < 0$, beyond the throat, the quantity $f(\Phi) = \Phi$ in the Lagrangian (1) is negative, which means that the effective gravitational constant G_{eff} is negative there.

The wormhole nature of this solution for $\omega = 0$ was pointed out in 1996 in [26], where its multidimensional generalization was also indicated.

2.4 General STT, $\varepsilon = -1$

If $\varepsilon = -1$ (so $l(\Phi) < 0$ and $\omega < -3/2$) there are three branches in the vacuum solution (10)–(12) according to the sign of k , see (9). They correspond to Brans classes II–IV [22]. Of interest for us is the case $k < 0$ (see [10, 27] for more complete descriptions). Then, as is easily verified, the metric (10) in \mathbb{M}_E [where now $s(k, u) = k^{-1} \sin ku$] has two flat spatial infinities at $u = 0$ and $u = \pi/|k|$ (choosing this half-wave of the sine function without loss of generality). It is the anti-Fisher wormhole, repeatedly described beginning with the papers [10]

and [28]. And it is also evident that the wormhole nature of this solution is preserved in *any* STT (1) in which the function $f(\Phi)$ is smooth and positive in the range $0 \leq u \leq \pi/|k|$. This fact was already pointed out in 1973 [10]. The BD theory is just a special case of such a theory, see Eq. (13) where, as before, $\phi = Cu$. Let us note that the value $\omega = -2$ is not distinguished in any way: wormhole solutions do appear for any $\omega < -3/2$, in the whole ghost range of the BD theory.

For full clarity, let us write this BD solution in a form in which its wormhole nature is more obvious. Substituting $|k|u = \cot^{-1}(x/|k|)$, we obtain

$$\begin{aligned} ds_J^2 &= e^{-(2m+\sigma)u} dt^2 - e^{(2m-\sigma)u} [dx^2 \\ &\quad + (k^2 + x^2) d\Omega^2], \\ \sigma &:= -C/(2|k|\sqrt{|\omega + 3/2|}), \\ 1 + m^2/k^2 &= \sigma^2 |2\omega + 3|, \end{aligned} \quad (19)$$

where x ranges over the whole real axis. Since $u \in (0, \pi/|k|)$, the exponentials in (19) do not affect the qualitative nature of the metric.

3 Wormholes in $F(R)$ gravity

In the above examples, we only discussed massless field configurations, with $U(\Phi) \equiv 0$. However, the theorem proved in [5] concerns the general case of the theory (1), with any potentials $U(\Phi)$. It therefore encompasses not only STT but also the metric $F(R)$ theories of gravity which are known to be equivalent to the BD theory with $\omega = 0$ and potentials $U(\Phi)$ whose form depends on the choice of the function $F(R)$ (see, e.g., the reviews [29, 30] and references therein). Indeed, the field equations

$$F_R R_\mu^\nu - \frac{1}{2} F \delta_\mu^\nu + (\nabla_\mu \nabla^\nu - \delta_\mu^\nu \square) F_R - T_\mu^\nu = 0 \quad (20)$$

($F_R := dF/dR$) obtained by variation of the Lagrangian

$$L_F = \frac{1}{2} F(R) + L_m \quad (21)$$

(where $F_{RR} := d^2 F/dR^2$ is not identically zero) with respect to the metric are equivalent to those due to the Lagrangian (1) with

$$\begin{aligned} f(\Phi) &= F_R, & h(\Phi) &= 0, \\ 2U(\Phi) &= R F_R - F, \end{aligned} \quad (22)$$

which is nothing else but the BD theory with $\omega = 0$, whose form (6) is obtained by introducing the

parametrization $f(\Phi) = \Phi$. It should be stressed that the two equivalent forms of the theory correspond to the same (Jordan) conformal frame and employ the same metric $g_{\mu\nu}$. Thus the condition $f(\Phi) > 0$ takes the form $dF/dR > 0$ in $F(R)$ gravity. The function $l(\Phi)$ (2) cannot become negative, however, it can reach zero at exceptional values of R where $F_{RR} = 0$ while $f = F_R \neq 0$. In the latter case, the parametrization $f(\Phi) = \Phi$ is no longer applicable since $f(\Phi)$ loses monotonicity.

Hence, according to [5], the only opportunity of obtaining wormhole solutions in the theory (21), both vacuum and supported by matter respecting the NEC, is connected with conformal continuations through a sphere at which $F(R)$ has an extremum reached at some value $R = R_0$. As is the case with STT, such spheres, being regular in the Jordan frame, are singular in the Einstein frame. After crossing such a sphere, one gets [31] to a region with negative values of Φ in the BD formulation of the same theory, which in turn correspond to negative values of the effective gravitational constant G_{eff} . It has been shown [11] that the wormhole behavior is generic among such special solutions. However, similarly to the case $f(\Phi_0) = 0$ in STT, in dynamic solutions there appears a generic spacelike anisotropic curvature singularity where the Kretschmann invariant diverges as $R \rightarrow R_0$ [33, 34].

The opportunity of $F_{RR} = 0$ while $F_R \neq 0$ at some $R = R_0$ is also of interest. At such surfaces, a weak singularity can arise at which the space-time metric and Riemann tensor components are finite (see [35] for a discussion of its structure in the cosmological case), but R generically behaves there as $R_0 + \lambda\sqrt{u - u_0}$, $\lambda = \text{const}$ if $u = u_0$ is the surface at which $R = R_0$, where u is the time coordinate in the cosmological setting or a spatial coordinate in a static space-time. One can check that the terms with derivatives of F_R in Eq. (20) remain finite at $u = u_0$ in this case. Still differential curvature invariants are infinite at such a surface, and there is no analytic extension across it. Solutions with $\lambda = 0$ which are regular at $R = R_0$ are not excluded but require strong fine tuning of initial or boundary conditions. However, in all such cases, according to [5], globally regular wormhole solutions cannot appear as long as F_R is everywhere positive.

Possible wormhole geometries in $F(R)$ gravity have been discussed in [16, 32]. In [32], only con-

ditions near a throat were discussed; we can once again recall that our results [5], based on conformal mappings, do not rule out the existence of throats. In [16] a few specific examples of wormhole solutions in $F(R)$ gravity, supported by matter with anisotropic pressure and without a fully specified equation of state, were found. However, in the first two examples, Eqs. (25), (31) of that paper, F_R is negative over some range of R for matter satisfying the weak energy condition. In the other two examples, Eqs. (35) and (39) of [16], either F_R becomes zero at the throat, so that G_{eff} is infinite and becomes negative beyond it, or the radial pressure and/or density of matter diverges there. It is once again confirmed that static wormhole-like solutions can formally exist in $F(R)$ theory but look rather unrealistic (cf. [36]).

4 Concluding remarks

One can ask a natural question: why, in rather a simple subject, quite a number of incorrect inferences appear in the literature. In our view, one reason is purely terminological: some authors call wormholes anything that has a throat. Actually their analysis reduces to proving its existence only. Meanwhile, to prove that there is a wormhole it is sufficient to find two (not necessarily flat) regular asymptotic regions with growing $r(u)$ (or a similar function if spherical symmetry is absent), and the existence of its minimum is then evident even without explicitly pointing it out.

Another reason for wrong or incomplete inferences is in some cases a very clumsy parametrization used in BD (and other) solutions. For instance, with the parameters C and λ mentioned in footnote 4 it is impossible to consider the case $a = \xi$ in the solution (14) which is of particular interest.

And one more oddity of some wormhole studies is a persistent usage of the curvature coordinate r , which is certainly convenient for solving the field equations in many (but not all) cases but is two-valued in the throat region. It really looks funny when a solution is first written in terms of an admissible coordinate u but is then transformed to r with much effort in order to seek a throat using the so-called shape function $b(r)$. Meanwhile, it is sufficient just to find a minimum of $r(u)$. Using r as a coordinate, one frequently remains restricted to one half of the configuration while another half, even if its metric is the same, can have different properties

(as, e.g., in the solution (18) where $G_{\text{eff}} < 0$ beyond the throat). In this way, one also sometimes loses wormhole solutions which are asymmetric with respect to the throat. Though, with certain care, asymmetric wormholes can certainly be described in the r parametrization as well [37].

We hope that these remarks can be of some use for researchers and especially students working in this field.

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References

- [1] D. Hochberg and M. Visser, Phys. Rev. D **56**, 4745 (1997) (arXiv:gr-qc/9704082).
- [2] D. Hochberg and M. Visser, Phys. Rev. D **58**, 044021 (1998) (arXiv:gr-qc/9802046).
- [3] S. A. Hayward, Phys. Rev. D **79**, 124001 (2009) (arXiv:0903.5438).
- [4] A. V. B. Arellano and F. S. N. Lobo, Class. Quantum Grav. **23**, 5811-5824 (2006).
- [5] K. A. Bronnikov and A. A. Starobinsky, JETP Lett. **85**, 1 (2007) (arXiv:gr-qc/0612032).
- [6] K. A. Bronnikov, S. B. Fadeev and A. V. Michtchenko, Gen. Rel. Grav. **35**, 505 (2003) (arXiv:gr-qc/0212065).
- [7] K. A. Bronnikov and A. A. Starobinsky, Mod. Phys. Lett. A **24**, 1559 (2009) (arXiv:0903.5173).
- [8] K. A. Bronnikov and M. S. Chernakova, Grav. Cosmol. **13**, 51 (2007) (arXiv:gr-qc/0703107).
- [9] K. A. Bronnikov, C. P. Constantinidis, R. L. Evangelista, and J. C. Fabris, Int. J. Mod. Phys. D **8**, 481 (1999) (arXiv:gr-qc/9903028).
- [10] K. A. Bronnikov, Acta Phys. Pol. B **32**, 3571 (2001) (arXiv:gr-qc/0110125).
- [11] K. A. Bronnikov, J. Math. Phys. **43**, 6096 (2002) (arXiv:gr-qc/0204001).
- [12] K. A. Bronnikov and S. Grinyok, Grav. Cosmol. **7**, 297 (2001) (arXiv:gr-qc/0201083).
- [13] A. A. Starobinsky, Sov. Astron. Lett. **7**, 36 (1981).
- [14] A. Bhattacharia, I. Nigmatzyanov, R. Izmailov and K. K. Nandi, Class. Quant. Grav. **26**, 235017 (2009) (arXiv:0910.1109).
- [15] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D **81**, 067501 (2010) (arXiv:1001.0995).
- [16] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D **80**, 104012 (2009) (arXiv:0909.5539).
- [17] K. A. Bronnikov and J. C. Fabris, Phys. Rev. Lett. **96**, 251101 (2006) (arXiv:gr-qc/0511109).
- [18] A. B. Balakin, J. P. S. Lemos, and A. E. Zayats, Phys. Rev. D **81**, 084015 (2010) (arXiv:1003.4584).
- [19] I. Z. Fisher, Zh. Eksp. Teor. Fiz. **18**, 636 (1948) (arXiv:gr-qc/9911008).
- [20] O. Bergmann and R. Leipnik, Phys. Rev. **107**, 1157 (1957).
- [21] I. Quiros, Phys. Rev. D **61**, 124026 (2000) (arXiv:gr-qc/9905071).
- [22] C. H. Brans, Phys. Rev. **125**, 2194 (1961).
- [23] A. G. Agnese and M. La Camera, Phys. Rev. D **51**, 2011 (1995).
- [24] V. Gorini, A. Yu. Kamenshchik, U. Moschella, V. Pasquier and A. A. Starobinsky, Phys. Rev. D **78**, 064064 (2008) (arXiv:0807.2740).
- [25] V. Gorini, A. Yu. Kamenshchik, U. Moschella, O. F. Piattella and A. A. Starobinsky, Phys. Rev. D **80**, 104038 (2009) (arXiv:0909.0866).
- [26] K. A. Bronnikov, Grav. Cosmol. **2**, 221 (1996) (arXiv:gr-qc/9703020).
- [27] K. A. Bronnikov, M. S. Chernakova, J. C. Fabris, N. Pinto-Neto and M. E. Rodrigues, Int. J. Mod. Phys. D **17**, 25 (2008) (arXiv:gr-qc/0609084).
- [28] H. G. Ellis, J. Math. Phys. **14**, 104 (1973).
- [29] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010) (arXiv:0805.1726).
- [30] A. De Felice and S. Tsujikawa, Living Rev. in Relativity, arXiv:1002.4928.
- [31] K. A. Bronnikov and M. S. Chernakova, Izv. Vuzov, Fizika 9, 46–51 (2005), Russ. Phys. J. **48**, 940 (2005) (arXiv:gr-qc/0503025).
- [32] N. Furey and A. DeBenedictis, Class. Quantum Grav. **22**, 313-322 (2006) (arXiv:gr-qc/0410088).
- [33] H. Nariai, Prog. Theor. Phys. **49**, 165 (1973).
- [34] V. Ts. Gurovich and A. A. Starobinsky, Sov. Phys. - JETP **50**, 844 (1979).
- [35] S. A. Appleby, R. A. Battye and A. A. Starobinsky, JCAP **1006**, 005 (2010) (arXiv:0909.1737).
- [36] D. H. Coule, Class. Quantum Grav. **10**, L25 (1993).
- [37] K. A. Bronnikov and Sung-Won Kim, Phys. Rev. D **67**, 064027 (2003) (arXiv:gr-qc/0212112).